Journal of Applied Econometrics and Statistics
Vol. 3, No. 1-2, 2024, pp. 61-78
https://DOI:10.47509/JAES.2024.v03i01-2.04

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

Numerical Estimation of the Weibull Distribution Parameters Using Adams's Method

M. Maswadah

Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt
E-mail: maswadah@hotmail.com

Received: 28 February 2024; Revised: 05 April 2024;
Accepted 13 April 2024; Publication: 30 December 2024

Abstract: Recently, numerical analysis has been used effectively for estimating the lifetime distribution parameters in the literature. Therefore, the main objective of this paper is to introduce a new numerical estimation technique, such as Adams's method. This method has been used for estimating the Weibull model parameters and comparing them to the Bayes estimations based on different priors via Monte Carlo simulations. The simulation results indicated that Adams's method is more efficient than Bayes' method. Finally, two real data sets have been analyzed for illustrations and to compare the proposed methods based on the generalized progressive hybrid censoring data.

Keywords: Bayesian estimation; Characteristic priors; Informative prior; Kernel prior.

INTRODUCTION

In statistical inference and reliability, numerical analysis has been used for estimating the distribution parameters using the Runge-Kutta method in the literature, see [20, 21, 22]. Thus, the main objective of this work is to introduce a new estimation method using a numerical technique such as Adams's method for estimating the distribution parameters and compare to the Bayesian estimates based on the characteristic, informative gamma, and kernel priors.

To illustrate that, we employed the proposed method on one of the most employed lifetime distributions and reliability theories, the Weibull distribution, for its flexibility in describing the lifetime variables of constant hazard rate as well as non-constant hazard rate, and it has been shown to be useful for modelling and analyzing lifetime data in medical, biological, and engineering sciences. Moreover, it is one of the models that has two well-known distributions as special cases, such as the exponential and Rayleigh distributions, and it has different priors for the parameters that have been extensively studied in the literature.

Many authors used the informative prior for the Weibull model parameters, among them, [24] derived an informative conjugate prior by assuming each of the parameters has a gamma distribution. [5] and [6] proposed a different prior based on the prior information on the reliability level or the hazard rate at a given time and converted it to information about the model parameters. [2] derived the estimations of the parameters based on the classical and Bayesian approaches. [13] presented the reliability and quantile analysis for the Weibull distribution. [3] derived the maximum likelihood estimation (MLE) for the Weibull model parameters based on complete and censored samples. [26] derived the MLEs for the Weibull model parameters based on type-II progressively censored samples. [25] applied the MLE and Bayes methods for estimating the Weibull parameters based on censored samples, [27] derived the parameter estimation based on progressively censored data, and [19] derived the empirical Bayes inference for the Weibull Model parameters. For a continuation of these efforts, the purpose of this paper is to derive the point estimations for the parameters based on generalized progressive hybrid censored samples using Adams's method and compare to the Bayes' method when the underlying distribution is the two-parameter Weibull distribution, which has a probability density function and cumulative distribution function as given respectively by:

$$f(x) = \alpha\beta x^{\alpha-1} \exp(-\beta x^{\alpha}),\ x > 0, \tag{1.1}$$

$$F(x) = 1 - \exp(-\beta x^{\alpha}),\ x > 0, \tag{1.2}$$

$\alpha, \beta > 0$ are the shape and scale parameters respectively.

In statistical analysis, the progressive censoring is the familiar schemes in both industrial life testing applications and clinical trials that allows the removal of surviving experimental units before the termination of the test. However, the disadvantages of the progressive type-II censoring scheme are that the time of the experiment can be very long if the units are highly reliable. experiments are often terminated before all units on the test fail due to cost or time considerations. Therefore, [13] recently proposed a censoring scheme called the Type-II progressively hybrid censoring scheme, with the disadvantage that very few failures may occur before time point T. To provide a guarantee of the number of failures observed as well as the time to complete the test, [8] and [9] proposed the generalized progressive hybrid-censoring scheme (GPHCS), which modifies the progressive hybrid censoring scheme. It allows the experiment to continue beyond time T to observe at least k failures if the number of failures is less than m. Thus, we have three cases for terminating the test: The first one is if the specified number of failures m is less than T. The second case is if T is less than m, where J is the number of observed units at the time T. The third case is if T is less than k. The GPHCS algorithm can be seen in [20, 21, 22].

Thus, given a generalized progressive hybrid censored sample, the likelihood function for the three different cases can be written in a unified form as follows:

$$L(\underline{x};0) = C \prod_{i=1}^{N} f(x_{i,m,N})[1-F(x_{i,m,N})]^{R_i}[1-F(T)]^{R_T^*\delta}, \tag{1.3}$$

$$N = \begin{cases} m, & \delta = 0, \quad if \quad X_{k:m:N} \le X_{m:m:N} < 1 \\ k, & \delta = 0, \quad if \quad T < X_{K:m:N} \le X_{m:m:N}, \\ J, & \delta = 1, \quad if \quad X_{k:m:N} < T < X_{m:m:N} \end{cases}$$

where R_T^* is the number of surviving units that are removed at the stopping time $T^* = \max\{X_{k:m:N}), \min\{X_{m:m:N}), T\}\}$.

The GPHCS has been applied for some distributions such as the Weibull distribution see [9], inverse Weibull distribution, see [20, 23], Exponential distribution, see [8], and [11], Rayleigh distribution, see [7], shape-scale family, see [21], and the generalized Weibull distribution, see [22].

2. ESTIMATION METHODS

Adams's Method

Theoretically, it is known that the traditional log likelihood function, $H(x, \theta)$, depends on the unknown parameter $\theta = (\alpha, \beta)$ and the data X. Thus, the MLE $\hat{\theta}$ of θ is the solution of the stationary equation $\frac{\partial H(\theta|X)}{\partial \theta} = 0$. Applying the implicit function theorem to the stationary equation with considering all partial derivatives as well as the total derivatives are assumed to be evaluated at some known value of $\theta = \theta^*$ or at any initial value θ_0. By taking the X-derivatives to the stationary equation we obtain

$$\frac{d}{dx}\left(\frac{\partial H(\theta|X)}{\partial \theta}\right) = \left.\frac{\partial^2 H(\theta|X)}{\partial \theta \partial x}\right|_{\theta=\theta^*} + \left.\frac{\partial^2 H(\theta|X)}{\partial \theta^2}\right|_{\theta=\theta^*} \frac{d\theta^*}{dx} = 0. \tag{2.4}$$

Solving (2.4) we obtain the first derivative of θ^* with respect to X at $\theta = \theta^*$ as:

$$\frac{d\theta^*}{dx} = -\left(\left.\frac{\partial^2 H(\theta|X)}{\partial \theta^2}\right|_{\theta=\theta^*}\right)^{-1} \left.\frac{\partial^2 H(\theta|X)}{\partial \theta\, \partial x}\right|_{\theta=\theta^*}. \tag{2.5}$$

Thus, we can write (2.5) as

$$\frac{d\theta^*}{dx} = f(x, \theta^*), \text{ at } \theta = \theta^*, \tag{2.6}$$

where $f(x, \theta^*) = -\left(\left.\frac{\partial^2 H(\theta|X)}{\partial\theta^2}\right|_{\theta=\theta^*}\right)^{-1} \left.\frac{\partial^2 H(\theta|X)}{\partial\theta\,\partial x}\right|_{\theta=\theta^*}$.

The equation (2.6) is first order ordinary differential equation in θ^*. A numerical method such as Adams's method can be used for finding the approximate solution given a trial set of parameter values and initial conditions, a procedure which is referred by engineers as simulations. If the initial conditions are unavailable, they must be appended to the parameter $\hat{\theta}$ as quantities with respect to which the fit is optimized. Thus, Adams recurrence solution for (2.6) can be obtained as:

$$\theta^*_{N+1} = \theta^*_N + h[23f(x_N, \theta^*) - 16f(x_{N-1}, \theta^*) + 5f(x_{N-2}, \theta^*)]/12,$$
$$\text{for } N = 0, 1, 2, \ldots, \tag{2.7}$$

Here h is a small known value (say, $1E-02$) and $\theta^*_0 = \theta_0$, is the initial value for θ^*.

The iterative process is continued using (2.7) until two consecutive numerical solutions are almost the same, that is if $|\theta^*_{N+1} - \theta^*_N| < 1E-05$. for $N = 0, 1, 2, 3, \ldots$.

Using α and β instead of θ in (2.7) we get the Adams estimator for each parameter, respectively.

2.2 BAYES METHOD

In this section, the Bayes estimations will be derived based on three different priors as follows:

I. Informative Prior

We consider the unknown parameters α and β have independent gamma prior distributions with joint probability density function defined as the following:

$$g(\alpha, \beta) \propto \alpha^{a-1}\, \beta^{c-1}\, e^{-d\,\beta - b\alpha)} \tag{2.8}$$

The hyper-parameter a, b, c and d are assumed to be known and positives to reflect the prior belief about the unknown parameters.

II. Kernel Prior

For deriving the kernel prior, we introduce the bivariate kernel density estimator for the unknown probability density function $g(\alpha, \beta)$ with support on $(0, \infty)$, which is defined as follows:

$$\hat{g}(\alpha, \beta) = \frac{1}{Nh_1h_2}\sum_{i=1}^{N} K\left(\frac{\alpha-\hat{\alpha}_i}{h_1}, \frac{\beta-\hat{\beta}_i}{h_2}\right), \tag{2.9}$$

h_i, $i = 1, 2$ are called the bandwidths or smoothing parameters, which are chosen such that $h_i \to 0$ and $Nh_i \to$ as $N \to \infty$, where N is the sample size. The influence of the smoothing parameter h is critical because it determines the amount of smoothing. However, the optimal choice for h_i, which minimizes the mean squared errors is given by $h_i = 1.06 S_i \mathrm{N}^{-0.2}$, where S_i the sample standard deviation. The optimal choice for the kernel function $K(., .)$ can be used as the bivariate standard normal distribution for the parameters α and β. Based on the properties of the MLEs of the parameters, which are converging in probability with the original parameters, the kernel prior estimate can be derived. It is worthwhile to mention that this kernel prior has been used for some distributions, see [1] and [16-18].

III. Characteristic Prior

The characteristic function CF is the Fourier transform of the cumulative distribution function CDF, and hence there is a one-to-one correspondence between the CF and the CDF. Thus, the CF is fully characterizing the distribution of the underlying random variable. Since the CF can be estimated using the empirical characteristic function ECF, which retains all the information in the sample, it plays an increasing and important role in econometrics and finance, see [10]. Thus, based on the CF for two random variables and its inversion formula, the probability density function for the characteristic prior for the parameters α and β has been derived in the Appendix A as follows:

$$\hat{h}(\alpha, \beta) = \frac{1}{4n\pi^2} \sum_{i=1}^{n} \frac{1}{\left|(a - \hat{\alpha}_i)\left(\hat{\beta}_i - \beta\right)\right|}. \tag{2.10}$$

The general joint prior using (2.8), (2.9) and (2.10) can be written as follows:

$$Q(\alpha, \beta) = g(\alpha, \beta) h(\alpha, \beta) q(\alpha, \beta)$$

Thus, based on Bayes' theorem the posterior density for the unknown parameters α and β can be derived using the likelihood function of the GPHCS (3) and the general joint prior as follows:

$$f(\alpha, \beta \mid \underline{X}) = KQ(\alpha, \beta) L(X; \alpha, \beta)$$

$$= K\hat{g}^{\,p_1}(\alpha)\hat{g}^{\,p_2}(\beta)\hat{q}^{\,l_1}(\alpha) q^{\,l_2}(\beta)\alpha^{N+a-1}\beta^{N+c-1} exp\left[-\alpha b + (\alpha - 1)\sum_{i=1}^{N} \ln(x_i)\right].$$

$$\times exp\left[-\beta\left(d + \sum_{i=1}^{N}(R_i + 1)x_i^{\alpha} + \delta R_T^{*} T^{\alpha}\right)\right.$$

(i) For the informative gamma prior $p_1 = p_2 = 0$, $l_1 = l_2 = 0$, as in (2.8).

(ii) For the kernel prior $p_1 = p_2 = 1$ and $a = c = 1$, $b = d = 0$, as in (2.9).

(iii) For the characteristic prior $p_1 = p_2 = 0$, $l_1 = l_2 = 1$ and $a = c = 1$, $b = d = 0$, as in (2.10).

The marginal posterior densities for the parameters based on the kernel and the characteristics priors cannot be solved analytically, but for the informative gamma prior can be evaluated as

$$g(\alpha \mid \underline{X}) = K\,\Gamma(N+c)\alpha^{N+1-1}\left[d + \sum_{i=1}^{N}(R_i+1)x_i^{\alpha} + \delta R_T^{*}T^{\alpha}\right]^{-(N+c)}$$

$$\exp\left[-\alpha b + (\alpha-1)\sum_{i=1}^{N}\ln(x_i)\right],$$

$$g(\beta \mid \underline{X}) = K\int_0^{\infty}\alpha^{N+a-1}\beta^{N+c-1}\exp\left[-\beta\left(d + \sum_{i=1}^{N}(R_i+1)x_i^{\alpha} + \delta R_T^{*}T^{\alpha}\right)\right]$$

$$\times\exp\left[-\alpha b + (\alpha-1)\sum_{i=1}^{N}\ln(x_i)\right],$$

K is the normalizing constant and can be derived as follows:

$$K^{-1} = \Gamma(N+c)\int_0^{\infty}\alpha^{(N+a-1)}\left[d + \sum_{i=1}^{N}(R_i+1)x_i^{\alpha} + \delta R_T^{*}T^{\alpha}\right]^{-(N+c)}$$

$$\times\exp\left[-\alpha b + (\alpha-1)\sum_{i=1}^{N}\ln(x_1)\right]d\alpha.$$

3. SIMULATION STUDY

The purpose of the simulation study is to compare the performance of the estimates using Adams and Bayes methods based on the informative gamma, the kernel and the characteristic priors, through two criteria the average bias (AVB) and the mean squared error (MSE) as given by:

$$AVB = \frac{1}{L}\sum_{i=1}^{L}\left|\hat{\theta}_i - \theta\right|, \text{ and } MSE = \sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2/L$$

$\hat{\theta}$ is the estimate of θ and L is the number of replications.

In our simulation study, we choose different values for the hyperparameters of α and β say: $(a, b, c, d) = (5, 3, 5, 3)$ and two values for the parameter $\theta = (1, 2)$, and two values for the parameter $\beta = (2, 3)$ respectively.

Using the above parameter values for generating different samples from the Weibull distribution with sizes n = 20, 40 and 60 to represent small, moderate and large sizes. To assess the performance of the parameter estimates using Adams and Bayes methods, the AVB and MSE for each one were calculated using 1000 replications.

From the simulation results in Tables [3-6], some of the points are quite clear based on these estimates and the others have been summarized in the following main points:

1. It is clear that the point estimates for the parameters based on Adams's method have the smallest estimated AVB and MSE values as compared with the estimates based on Bayes' method using three different priors.
2. In general, for the parameters α and β, the estimated MSE values based on the kernel prior are often less than the estimated based on the informative gamma prior and almost close to the ones for Adams's method.
3. The estimated MSE values increase as the value of α increases and decrease as the value of β increases.
4. The estimated MSE values decrease as the hyperparameters of the informative prior decrease.

   The estimated MSE values for the parameters decrease as the sample sizes and the termination time of the experiment T increase as expected.

As a conclusion, it appears that the point estimates based on Adams's method compete and outperform Bayes' method using the different priors.

4. REAL DATA ANALYSIS

In this section, we studied two real datasets to study the performance of the proposed methods on the Weibull model, which is the most desirable and widely used lifetime distribution. This distribution has been used in many applications in various fields and in new areas such as biomedical science and survival analysis to describe the lifetime of specific mortality and failure rates. Hence, we have fitted these datasets using some goodness of fit tests such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Chi-Square (CH2) tests for a significance level equals to 0.05.

4.1 Ball Bearings Data Application

In this section we consider one real dataset from [14] and [15]. The data arose in tests on the endurance of deep groove ball bearings. The data are the number of million revolutions before failure for each of the 23 ball bearings in the life test and they are as follows:

17.88 28.92 33.00 41.52 42.12 45.60 48.40 51.84 51.96 54.12 55.56 67.80 68.64 68.64 68.88 84.12 93.12 98.64 105.12 105.84 127.92 128.04 173.40.

Figure 1a indicates these data are good fit for the Weibull distribution. Also, in Table 1, we can see the calculated values for the test statistics are less than the critical values and the power of the tests are greater than the significance level 0.05.

We noticed that Adams and Bayes estimates for ? are close to 2.5, which indicates that the above dataset is moderately bell shaped, which means slightly decreasing the number of revolutions of the ball bearings before failure, see Figure (1b). Also, Adams and Bayes estimates for ? are almost close to zero, which ensures this dataset is almost symmetric even with increasing time. Thus, this dataset ensures the strength of the ball bearings.

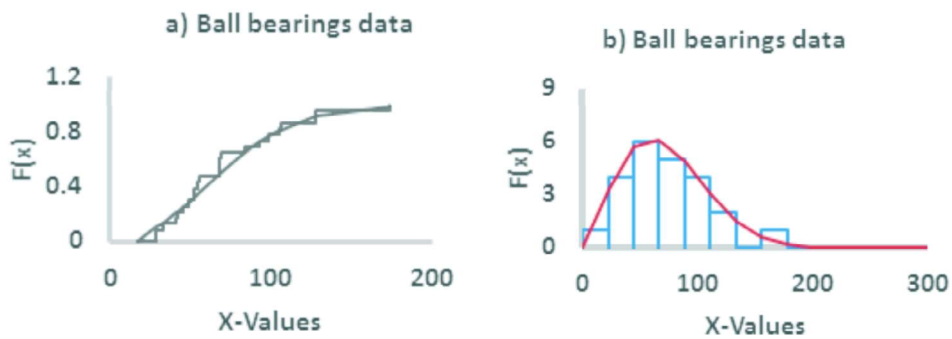

Fig. 1: (a) The Empirical CDF and the fitted CDF for the ball bearings data. (b) The Histogram and the fitted PDF for the ball bearings data.

4.2 Vinyl Chloride Data Application

As vinyl chloride is a known human carcinogen, exposure to this compound should be avoided as far as practicable, and levels should be kept as low as technically feasible. It is known that a concentration of vinyl chloride in drinking-water of 0.5 mg/litre was calculated to be associated with an excess risk of liver and brain tumors for exposure beginning in adulthood, and it would double the cancer risk for continuous exposure from birth. Therefore, we consider the dataset used by [4], which represents 34 data points in mg/L from the vinyl chloride that was obtained from clean upgrade monitoring wells, as:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

We found the Weibull model to be a good fit for this dataset, as shown in Table 1 and Figure (1a). For studying the concentration of vinyl chloride in the water of these wells based on this dataset, we find the estimates for the parameters, which represent the scale and shape of the concentration, using our model to determine the average concentration in the water. We noticed that Adams and Bayes estimates for α are close to 1.3, which indicates that the above dataset is moderately right-skewed, which means the concentration decreases with increasing time, see Figure (1 b). Also, Adams and Bayes estimates for β are close to 0.5, which ensures the dataset is right-skewed and the vinyl chloride concentration will decrease with increasing time, and therefore monitoring these wells is very significant.

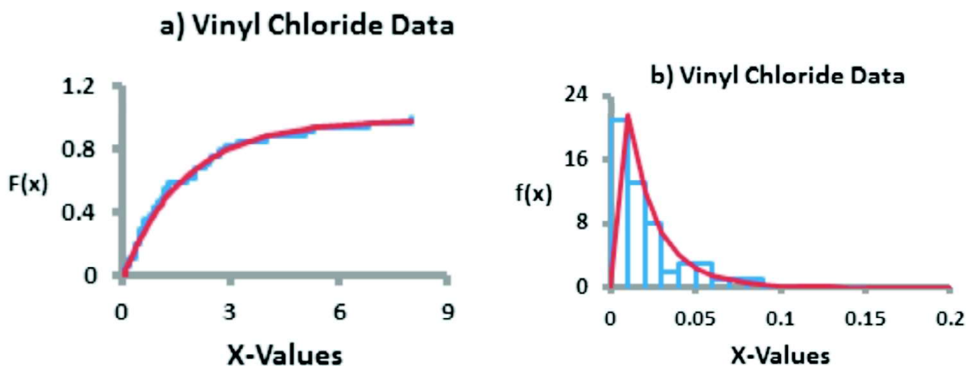

Fig. 2: (a) The Empirical CDF and the fitted CDF for the Vinyl Chloride data. (b) The Histogram and the fitted PDF for the Vinyl Chloride data.

Table 1: The critical and calculated values for the K-S, A-D and CH2 tests and their powers (p- values).

Data	*The Tests*	*Critical Value*	*Calculated Value*	*The P-values*	$\hat{\alpha}$	$\hat{\beta}$
The Ball Bearing Data, N = 23	K-S	0.8541	0.7239	0.2080	2.1015	9.5E-05
	A-D	0.7468	0.3283	0.5733		
	CH2	13.2922	2.9031	0.6098		
The Vinyl Chloride Data, N = 34	K-S	0.8624	0.5355	0.6525	1.0102	0.5263
	A-D	0.7504	0.2826	0.6708		
	CH2	15.428	4.9912	0.4474		

The results in Table 1 indicate that the Weibull model is a good fit for these datasets.The power of the tests is greater than the significance level of the tests, as shown in the Figures (1a, 2a) and the calculated values of the goodness of fit tests are smaller than the critical values of the tests. Thus, the results in Table 2 indicated that the estimated MSE values based on Adams's method are smaller than those based on Bayes' method for large values of T when considering the MLE values as the true values of the parameters. Thus, the results of these data sets ensure the simulation results.

Table 2: The estimate and the root mean squared errors (MSEs) for the parameters α and β based on the Picard and Bayes methods using gamma and Kernel priors based on the GHPCS: for $m = n/2$, and $k = m/2$ with $a = 5$, $b = 3$, $c = 5$, $d = 3$

			Adams	*Estamite*	*Gamma*	*Prior*	*Kernel*	*Prior*
Samples	*T*	*Parameters*	*Estimate*	*MSE*	*Estimate*	*MSE*	*Estimate*	*MSE*
The Ball	50		2.0586	0.00179	2.4574	0.0432	2.2743	0.01236
Bearings			0.00765	5.7E-05	0.06048	0.003645	0.05117	0.002609
Data,	120		2.0585	0.00180	2.5473	0.0432	2.3532	0.0362
N = 23			0.005528	2.9E-05	0.0625	0.00389	0.05082	0.002573
The vinyl	0.75		0.9769	0.001106	1.3192	0.0956	0.8982	0.01254
Chloride								
Data,	1.25		0.9767	0.001122	1.3326	0.1039	0.9007	0.01204
N = 34			0.4810	0.002044	0.3568	0.02873	0.3062	0.04842

CONCLUSIONS

It is known that Bayes estimation method based on the informative gamma prior is more efficient than most of the estimation methods in reliability theory despite its subjectivity to information other than data. Thus, in this work, we transformed the stationary equation into a differential equation, which can be solved numerically with any iteration techniques such as Adams's method. We found that, the parameter estimates based on Adams's method are more efficient than those based on Bayes' method using the three different priors such as the informative gamma, characteristic and kernel priors based on the generalized progressive hybrid censoring scheme.

Conflict of Interest

The author declares no conflict of interest.

REFERENCES

1. Ahsanullah, M., Maswadah, M. and Seham, M. (2013). Kernel inference on the generalized gamma distribution based on generalized order statistics. *J. Stat. Theory Appl.* Vol. 12, pp. 152–172.
2. Abd-Elrahman, A.M. and Sultan, K.S. (2007). Reliability estimation based on general progressive censored data from the Weibull model: Comparison between Bayesian and Classical approaches. *METRON-International Journal of Statistics.* LX Vol. 2, pp. 239-257.
3. Balakrishnan N. and Kateri M. (2008). On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data. *Statistics and Probability Letters Journal.* Vol. 78, pp. 2971-2975.
4. Bhaumik, D.K. Kapur, K. and Gibbons, R.D. (2009). Testing parameters of a gamma distribution for small samples. Technometrics. Vol. 51, pp. 326–334.
5. Calabria, R. and Pulcini, G. (1994). An engineering approach to Bayes estimation for the Weibull distribution. Microelectron Reliability, Vol. 34, pp. 789-802.

6. Calabria, R. and Pulcini, G. (1995). Bayes prediction of number of failures in Weibull samples. Commun. Statist.-Theory Meth. Vol. 24(2), pp. 487-499.
7. Cho, Y., Sun, H., Lee, K. (2014). An estimation of the entropy for a Rayleigh distribution based on doubly generalized Type-II hybrid censored samples. Entropy, Vol. 16, pp. 3655–3669.
8. Cho, Y., Sun, H., Lee, K. (2015 a). Exact likelihood inference for an exponential parameter under generalized progressive hybrid censoring scheme. Stat. Methods., Vol. 23, pp. 18–34.
9. Cho, Y., Sun, H., Lee, K. (2015 b). Estimating the entropy of a Weibull distribution under generalized progressive hybrid censoring. Entropy 17, pp. 102–122.
10. Feuerverger, F. (1990). An efficiency result for the empirical characteristic function in stationary time series models. The Canadian Journal of Statistics. Vol.18, pp. 155-161.
11. Gorny, J, and Cramer, E. (2016). Exact likelihood inference for exponential distribution under generalized progressive hybrid censoring schemes. Stat. Methodology. 29 (2016), pp. 70-94.
12. Khan, M.S., Pasha, G.R. and Pasha, A.H. (2007). Reliability and quantile analysis of the Weibull distribution. *Journal of Statistics*, Vol. 14, pp. 32-52.
13. Kundu, D., Joarder, A. (2006) Analysis of Type-II progressively hybrid censored data. Comput. Stat Data Anal. Vol. 50, pp. 2509–2528.
14. Kundu, D. and Pradhan, B. (2009). On progressively censored generalized exponential distribution. Test, 18, P. 497-515.
15. Lawless, J.F. (1982). Statistical Models and Methods for Lifetime Data. John Wiley & Sons. New York.
16. Maswadah, M. (2006). Kernel inference on the inverse Weibull distribution. The Korean Communications in Statistics, Vol. 13(3): pp. 503-512.
17. Maswadah, M, (2007). Kernel inference on the Weibull distribution. Proc. Of the Third National Statistical Conference. Lahore, Pakistan, Vol.14, pp. 77-86.
18. Maswadah, M. (2010). Kernel inference on the type-II Extreme value distribution. Proceedings of the Tenth Islamic Countries Conference on Statistical Sciences (ICCS-X), Lahore, Pakistan, Vol. II: pp. 870-880.
19. Maswadah, M. (2013). Empirical Bayes Inference for the Weibull Model. *The Computational Statistics Journal,* Vol. 28(3), pp. 2849-2859. Comput. Stat. DOI 10.1007/s00180-013-0438-1.
20. Maswadah, M. (2021). An optimal point estimation method for the inverse Weibull model parameters using the Runge-Kutta method, *Aligarh Journal of Statistics* (AJS), Vol. 41, pp.1-21.
21. Maswadah, M. (2022). Improved maximum likelihood estimation of the shape-scale family based on the generalized progressive hybrid censoring scheme. *Journal of Applied Statistics*, Vol. 49(11), pp. 4825-4849. DOI: 10.1080/02664763.2021.1924638.
22. Maswadah, M. (2023). Numerical estimation method for the generalized Weibull distribution parameters. *International Journal of Applied Mathematics, Computational science, and system engineering* (WSEAS). DOI: 10.37394/232026.2023.5.1.
23. Mohie El-Din, M.M. and Nagy, M. (2017). Estimation for Inverse Weibull distribution under Generalized Progressive Hybrid Censoring Scheme, *J. Stat. Appl. Pro. Lett.*, Vol. 4(3), pp. 97-107.

24. Nigm, A.M. (1989). An informative Bayesian inference of the Weibull life distribution. Commun. Statist.-Theory Meth. Vol. 18(3), pp. 897-911.
25. Panahi, H. and Asadi, S. (2011). Estimation of the Weibull distribution based on type-II censored samples. Applied Mathematical Sciences, Vol. 5(52), pp. 2549-2558.
26. Sahran, A.M. and AL-R Uzaizaa, A. (2010). Statistical inference in connection with the Weibull model using type-II progressively censored data with random scheme. Pak. J. Statist. Vol. 26 (1). pp. 267-279.
27. Wu, S. J. (2002). Estimation of the parameters of the Weibull distribution with progressively censored data. *J. Japan Statist. Soc.* Vol. 32(2). pp. 155-163.

for citation :

M. Maswadah (2024). Numerical Estimation of the Weibull Distribution Parameters Using Adams's Method. *Journal of Applied Econometrics and Statistics,* Vol. 3, No. 1-2, pp. 61-78.

Table 3: The Average bias (ABS) and the Mean Squared Errors (MSEs) in parentheses for the parameter α using Adams and Bayes methods with m = (n/2 and 3n/4) and k = (m/2 and 3m/4) at T = 0.75

						Bayes Estimations		
N	*m*	*k*	α	β	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
20	10	5	1	2	0.0572 (0.0052)	0.1203 (0.0147)	0.2062 (0.0760)	0.1507 (0.0363)
				3	0.0741 (0.0090)	0.1189 (0.0144)	0.1501 (0.0349)	0.1314 (0.0264)
			2	2	0.4281 (0.0154)	0.2568 (0.0662)	0.4145 (0.3100)	0.2109 (0.0701)
				3	0.2490 (0.2333)	0.2638 (0.0697)	0.2945 (0.1411)	0.2196 (0.0747)
		8	1	2	0.0600 (0.0057)	0.1208 (0.0149)	0.2015 (0.0714)	0.1499 (0.0355)
				3	0.0688 (0.0069)	0.1192 (0.0145)	0.1461 (0.0323)	0.1292 (0.0250)
			2	2	0.2154 (0.0590)	0.2550 (0.0654)	0.4045 (0.2946)	0.2086 (0.0666)
				3	0.2195 (0.0653)	0.2640 (0.0698)	0.2874 (0.1327)	0.2107 (0.0672)
	15	8	1	2	0.0584 (0.0056)	0.1211 (0.0149)	0.2099 (0.0819)	0.1534 (0.0384)
				3	0.0635 (0.0057)	0.1213 (0.0149)	0.1454 (0.0333)	0.1283 (0.0254)
			2	2	0.2115 (0.0609)	0.2539 (0.0651)	0.4481 (0.3570)	0.2162 (0.0710)
				3	0.2213 (0.0747)	0.2644 (0.0700)	0.2905 (0.1303)	0.2109 (0.0676)
		11	1	2	0.0538 (0.0044)	0.1231 (0.0154)	0.1918 (0.0658)	0.1377 (0.0308)
				3	0.0629 (0.0056)	0.1230 (0.0154)	0.1433 (0.0316)	0.1273 (0.0246)
			2	2	0.1995 (0.0493)	0.2571 (0.0704)	0.4251 (0.3359)	0.2044 (0.0653)
				3	0.2081 (0.0515)	0.2633 (0.0696)	0.2866 (0.1355)	0.2038 (0.0632)
40	20	10	1	2	0.0595 (0.0047)	0.1088 (0.0121)	0.1476 (0.0370)	0.1136 (0.0211)
				3	0.0681 (0.0056)	0.1114 (0.0125)	0.1138 (0.0193)	0.1022 (0.0155)
			2	2	0.2139 (0.0523)	0.2463 (0.0615)	0.3082 (0.1589)	0.1830 (0.0519)
				3	0.2243 (0.0568)	0.2588 (0.0671)	0.2238 (0.0767)	0.1857 (0.0527)
		15	1	2	0.0586 (0.0045)	0.1094 (0.0122)	0.1461 (0.0339)	0.1110 (0.0191)
				3	0.0668 (0.0053)	0.1113 (0.0125)	0.1107 (0.0189)	0.0993 (0.0151)
			2	2	0.2107 (0.0508)	0.2448 (0.0610)	0.3177 (0.1596)	0.1770 (0.0500)
				3	0.2213 (0.0541)	0.2582 (0.0669)	0.2164 (0.0720)	0.1831 (0.0500)
	30	15	1	2	0.0568 (0.0041)	0.1092 (0.0122)	0.1439 (0.0354)	0.1084 (0.0191)
				3	0.0655 (0.0052)	0.1127 (0.0128)	0.1088 (0.0181)	0.0985 (0.0148)
			2	2	0.2057 (0.0482)	0.2436 (0.0607)	0.3357 (0.1906)	0.1840 (0.0527)
				3	0.2148 (0.0507)	0.2577 (0.0667)	0.2139 (0.0726)	0.1764 (0.0481)
		23	1	2	0.0524 (0.0035)	0.1105 (0.0127)	0.1366 (0.0319)	0.1054 (0.0182)
				3	0.0615 (0.0045)	0.1137 (0.0131)	0.1013 (0.0158)	0.0947 (0.0136)
			2	2	0.1914 (0.0419)	0.2475 (0.0645)	0.2990 (0.1517)	0.1763 (0.0475)
				3	0.2080 (0.0472)	0.2544 (0.0655)	0.1982 (0.0628)	0.1627 (0.0410)
60	30	15	1	2	0.0551 (0.0036)	0.1024 (0.0109)	0.1189 (0.0236)	0.0912 (0.0130)
				3	0.0632 (0.0045)	0.1071 (0.0116)	0.0919 (0.0130)	0.0859 (0.0110)
			2	2	0.2060 (0.0456)	0.2382 (0.0586)	0.2557 (0.1106)	0.1569 (0.0379)
				3	0.2167 (0.0503)	0.2543 (0.0649)	0.1902 (0.0563)	0.1677 (0.0417)

						Bayes Estimations		
N	*m*	*k*	α	β	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
		23	1	2	0.0558 (0.0036)	0.1026 (0.0109)	0.1171 (0.0224)	0.0896 (0.0125)
				3	0.0630 (0.0045)	0.1070 (0.0115)	0.0936 (0.0132)	0.0869 (0.0113)
			2	2	0.2096 (0.0478)	0.2393 (0.0590)	0.2647 (0.1168)	0.1684 (0.0426)
				3	0.2140 (0.0486)	0.2530 (0.0644)	0.1825 (0.0512)	0.1613 (0.0388)
	45	23	1	2	0.0563 (0.0038)	0.1028 (0.0110)	0.1250 (0.0250)	0.0948 (0.0140)
				3	0.0645 (0.0047)	0.1070 (0.0115)	0.0957 (0.0139)	0.0884 (0.0117)
			2	2	0.2087 (0.0474)	0.2389 (0.0592)	0.2698 (0.1231)	0.1636 (0.0416)
				3	0.2144 (0.0490)	0.2535 (0.0646)	0.1824 (0.0511)	0.1626 (0.0393)
		34	1	2	0.0512 (0.0031)	0.1018 (0.0111)	0.1170 (0.0224)	0.0885 (0.0128)
				3	0.0592 (0.0040)	0.1071 (0.0116)	0.0822 (0.0106)	0.0786 (0.0097)
			2	2	0.1903 (0.0394)	0.2335 (0.0594)	0.2538 (0.1067)	0.1510 (0.0352)
				3	0.2104 (0.0470)	0.2471 (0.0621)	0.1749 (0.0458)	0.1539 (0.0354)

Table 4: The Average bias (ABS) and the Mean Squared Errors (MSEs) in parentheses for the parameter α using Adams and Bayes methods at T = 1.5 with m = (n/2 and 3n/4) and k = (m/2 and 3m/4)

						Bayes Estimations		
N	*m*	*k*	α	β	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
20	10	5	1	2	0.0553 (0.0048)	0.1230 (0.0154)	0.2068 (0.0810)	0.1496 (0.0371)
				3	0.0657 (0.0063)	0.1211 (0.0149)	0.1427 (0.0306)	0.1269 (0.0241)
			2	2	0.2010 (0.0515)	0.2520 (0.0646)	0.4633 (0.3775)	0.2084 (0.0679)
				3	0.2215 (0.0630)	0.2637 (0.0697)	0.2891 (0.1349)	0.2122 (0.0695)
		8	1	2	0.0541 (0.0045)	0.1224 (0.0152)	0.1954 (0.0680)	0.1394 (0.0315)
				3	0.0629 (0.0056)	0.1223 (0.0152)	0.1419 (0.0295)	0.1262 (0.0232)
			2	2	0.2111 (0.0597)	0.2541 (0.0653)	0.4410 (0.3468)	0.2130 (0.0702)
				3	0.2145 (0.0567)	0.2634 (0.0695)	0.2758 (0.1224)	0.2014 (0.0624)
	15	8	1	2	0.0524 (0.0043)	0.1234 (0.0155)	0.1959 (0.0701)	0.1416 (0.0330)
				3	0.0626 (0.0055)	0.1224 (0.0152)	0.1441 (0.0316)	0.1281 (0.0247)
			2	2	0.2097 (0.0602)	0.2556 (0.0670)	0.4295 (0.3431)	0.2088 (0.0677)
				3	0.2203 (0.0601)	0.2639 (0.0698)	0.2797 (0.1272)	0.2097 (0.0678)
		11	1	2	0.0516 (0.0043)	0.1244 (0.0158)	0.2005 (0.0724)	0.1438 (0.0340)
				3	0.0601 (0.0049)	0.1227 (0.0153)	0.1362 (0.0285)	0.1210 (0.0222)
			2	2	0.1952 (0.0480)	0.2570 (0.0683)	0.4395 (0.3586)	0.2107 (0.0672)
				3	0.2153 (0.0569)	0.2630 (0.0694)	0.2874 (0.1350)	0.2077 (0.0661)
40	20	10	1	2	0.0542 (0.0037)	0.1099 (0.0124)	0.1438 (0.0343)	0.1079 (0.0184)
				3	0.0625 (0.0046)	0.1129 (0.0129)	0.1058 (0.0171)	0.0978 (0.0144)
			2	2	0.2035 (0.0468)	0.2450 (0.0617)	0.3081 (0.1594)	0.1796 (0.0481)
				3	0.2124 (0.0502)	0.2565 (0.0662)	0.2054 (0.0667)	0.1694 (0.0445)

						Bayes Estimations		
N	m	k	α	β	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
		15	1	2	0.0553 (0.0039)	0.1099 (0.0124)	0.1468 (0.0353)	0.1106 (0.0193)
				3	0.0633 (0.0047)	0.1132 (0.0129)	0.1043 (0.0162)	0.0963 (0.0137)
			2	2	0.2044 (0.0476)	0.2453 (0.0632)	0.3066 (0.1577)	0.1756 (0.0471)
				3	0.2118 (0.0486)	0.2568 (0.0663)	0.2057 (0.0642)	0.1738 (0.0442)
	30	15	1	2	0.0532 (0.0036)	0.1113 (0.0130)	0.1360 (0.0321)	0.1040 (0.0178)
				3	0.0621 (0.0046)	0.1135 (0.0130)	0.1033 (0.0166)	0.0963 (0.0142)
			2	2	0.1953 (0.0446)	0.2430 (0.0616)	0.3159 (0.1672)	0.1751 (0.0485)
				3	0.2107 (0.0491)	0.2547 (0.0657)	0.2047 (0.0694)	0.1710 (0.0446)
		23	1	2	0.0501 (0.0032)	0.1112 (0.0132)	0.1386 (0.0331)	0.1056 (0.0184)
				3	0.0600 (0.0043)	0.1134 (0.0130)	0.1029 (0.0164)	0.0961 (0.0142)
			2	2	0.1885 (0.0403)	0.2469 (0.0643)	0.2865 (0.1382)	0.1692 (0.0440)
				3	0.2080 (0.0476)	0.2530 (0.0648)	0.2065 (0.0670)	0.1699 (0.0435)
60	30	15	1	2	0.0539 (0.0034)	0.1047 (0.0125)	0.1217 (0.0251)	0.0916 (0.0139)
				3	0.0601 (0.0040)	0.1074 (0.0117)	0.0855 (0.0112)	0.0815 (0.0100)
			2	2	0.2015 (0.0438)	0.2353 (0.0578)	0.2456 (0.1011)	0.1479 (0.0339)
				3	0.2122 (0.0474)	0.2514 (0.0638)	0.1661 (0.0432)	0.1511 (0.0351)
		23	1	2	0.0542 (0.0035)	0.1027 (0.0111)	0.1169 (0.0220)	0.0891 (0.0125)
				3	0.0605 (0.0041)	0.1076 (0.0117)	0.0859 (0.0112)	0.0818 (0.0100)
			2	2	0.2029 (0.0449)	0.2407 (0.0608)	0.2464 (0.1035)	0.1555 (0.0373)
				3	0.2121 (0.0478)	0.2484 (0.0625)	0.1786 (0.0484)	0.1591 (0.0375)
	45	23	1	2	0.0499 (0.0029)	0.1038 (0.0116)	0.1101 (0.0204)	0.0872 (0.0123)
				3	0.0595 (0.0040)	0.1065 (0.0116)	0.0830 (0.0106)	0.0797 (0.0097)
			2	2	0.1905 (0.0397)	0.2370 (0.0593)	0.2359 (0.0914)	0.1482 (0.0343)
				3	0.2088 (0.0462)	0.2457 (0.0617)	0.1770 (0.0480)	0.1555 (0.0360)
	34	1	2	0.0508 (0.0030)		0.1046 (0.0118)	0.1092 (0.0202)	0.0866 (0.0120)
			3	0.0591 (0.0040)		0.1061 (0.0115)	0.0830 (0.0109)	0.0792 (0.0098)
			2	2	0.1866 (0.0381)	0.2376 (0.0622)	0.2296 (0.0893)	0.1461 (0.0334)
				3	0.2071 (0.0458)	0.2462 (0.0623)	0.1714 (0.0472)	0.1526 (0.0352)

Table 5: The Average bias (ABS) and the Mean Squared Errors (MSEs) in parentheses for the parameter α using Adams and Bayes methods with m = (n/2 and 3n/4) and k = (m/2 and 3m/4) at T = 0.75

						Bayes Estimations		
N	*m*	*k*	α	β	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
20	10	5	1	2	0.1370 (0.0189)	0.3662 (0.1921)	0.3476 (0.1610)	0.1699 (0.0551)
				3	0.3096 (0.0958)	0.7344 (0.6455)	0.7127 (0.5614)	0.5883 (0.3633)
			2	2	0.2842 (0.0826)	0.3822 (0.2073)	0.4464 (0.2585)	0.1945 (0.0654)
				3	0.4940 (0.2507)	0.6452 (0.5575)	0.5509 (0.3758)	0.5182 (0.2917)
		8	1	2	0.1339 (0.0179)	0.3697 (0.1945)	0.3436 (0.1601)	0.1792 (0.0604)
				3	0.2785 (0.0775)	0.6858 (0.5674)	0.6571 (0.4743)	0.5547 (0.3218)
			2	2	0.2668 (0.0726)	0.3562 (0.1888)	0.4812 (0.2980)	0.1893 (0.0628)
				3	0.4704 (0.2260)	0.6162 (0.5104)	0.4858 (0.3040)	0.4901 (0.2631)
	15	8	1	2	0.1267 (0.0161)	0.3536 (0.1779)	0.3609 (0.1723)	0.1828 (0.0573)
				3	0.2543 (0.0647)	0.6850 (0.5680)	0.6230 (0.4398)	0.5377 (0.3077)
			2	2	0.2691 (0.0739)	0.3803 (0.2054)	0.4867 (0.3046)	0.1939 (0.0649)
				3	0.4706 (0.2266)	0.6223 (0.5244)	0.4924 (0.3179)	0.4934 (0.2697)
		11	1	2	0.1112 (0.0124)	0.3298 (0.1581)	0.3785 (0.1920)	0.1915 (0.0603)
				3	0.2190 (0.0480)	0.5918 (0.4315)	0.5336 (0.3358)	0.4845 (0.2552)
			2	2	0.2352 (0.0561)	0.3462 (0.1739)	0.5044 (0.3382)	0.2037 (0.0688)
				3	0.4087 (0.1697)	0.5164 (0.3802)	0.3614 (0.1957)	0.4285 (0.2094)
40	20	10	1	2	0.1405 (0.0198)	0.3260 (0.1557)	0.3563 (0.1705)	0.1917 (0.0616)
				3	0.2929 (0.0858)	0.6338 (0.4916)	0.5820 (0.3906)	0.5217 (0.2915)
			2	2	0.2891 (0.0846)	0.3388 (0.1666)	0.4497 (0.2620)	0.1885 (0.0602)
				3	0.5050 (0.2588)	0.6124 (0.4977)	0.4881 (0.3133)	0.4750 (0.2500)
		15	1	2	0.1385 (0.0192)	0.3135 (0.1448)	0.3592 (0.1754)	0.1836 (0.0582)
				3	0.2929 (0.0858)	0.6282 (0.4800)	0.5743 (0.3790)	0.5162 (0.2846)
			2	2	0.2744 (0.0761)	0.3205 (0.1529)	0.4573 (0.2766)	0.1917 (0.0624)
				3	0.4754 (0.2285)	0.5746 (0.4355)	0.4393 (0.2647)	0.4532 (0.2297)
	30	15	1	2	0.1368 (0.0187)	0.3105 (0.1425)	0.3640 (0.1798)	0.1852 (0.0569)
				3	0.2784 (0.0775)	0.6096 (0.4530)	0.5449 (0.3454)	0.4985 (0.2672)
			2	2	0.2756 (0.0769)	0.3409 (0.1706)	0.4678 (0.2867)	0.1927 (0.0646)
				3	0.4791 (0.2320)	0.5957 (0.4620)	0.4363 (0.2635)	0.4557 (0.2320)
		23	1	2	0.1083 (0.0117)	0.2858 (0.1192)	0.3621 (0.1845)	0.1955 (0.0580)
				3	0.2121 (0.0450)	0.4848 (0.2915)	0.4007 (0.2021)	0.4074 (0.1863)
			2	2	0.2266 (0.0517)	0.2863 (0.1240)	0.4167 (0.2496)	0.1969 (0.0604)
				3	0.3930 (0.1555)	0.4419 (0.2707)	0.2663 (0.1226)	0.3570 (0.1534)
60	30	15	1	2	0.1336 (0.0179)	0.2958 (0.1257)	0.3487 (0.1709)	0.1839 (0.0526)
				3	0.2697 (0.0727)	0.5436 (0.3627)	0.4636 (0.2622)	0.4533 (0.2258)
			2	2	0.2738 (0.0756)	0.3107 (0.1400)	0.4264 (0.2534)	0.1951 (0.0602)
				3	0.4919 (0.2440)	0.5798 (0.4366)	0.3996 (0.2310)	0.4354 (0.2141)

						Bayes Estimations		
N	*m*	*k*	*α*	*β*	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
		23	1	2	0.1296 (0.0168)	0.2800 (0.1124)	0.3400 (0.1633)	0.1779 (0.0495)
				3	0.2697 (0.0727)	0.5510 (0.3690)	0.4636 (0.2591)	0.4541 (0.2252)
			2	2	0.2708 (0.0740)	0.3154 (0.1444)	0.4334 (0.2574)	0.1958 (0.0615)
				3	0.4737 (0.2262)	0.5622 (0.4106)	0.3672 (0.2047)	0.4187 (0.2009)
	45	23	1	2	0.1356 (0.0184)	0.2964 (0.1278)	0.3642 (0.1803)	0.1926 (0.0619)
				3	0.2784 (0.0775)	0.5627 (0.3880)	0.4867 (0.2855)	0.4687 (0.2400)
			2	2	0.2718 (0.0744)	0.2983 (0.1308)	0.4318 (0.2629)	0.1870 (0.0557)
				3	0.4734 (0.2259)	0.5613 (0.4096)	0.3723 (0.2034)	0.4220 (0.2021)
		34	1	2	0.1093 (0.0119)	0.2560 (0.0964)	0.3286 (0.1559)	0.1798 (0.0489)
				3	0.2143 (0.0459)	0.4467 (0.2500)	0.3383 (0.1470)	0.3689 (0.1540)
			2	2	0.2283 (0.0524)	0.2591 (0.0984)	0.3663 (0.2110)	0.1780 (0.0474)
				3	0.3955 (0.1572)	0.4233 (0.2533)	0.2497 (0.1165)	0.3368 (0.1440)

Table 6: The Average bias (ABS) and the Mean Squared Errors (MSEs) in parentheses for the parameter α using Adams and Bayes methods with $m = (n/2 \ \& \ 3n/4)$ and $k = (m/2 \text{ and } 3m/4)$ at $T = 1.5$

						Bayes Estimations		
N	*m*	*k*	*α*	*β*	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
20	10	5	1	2	0.0976 (0.0096)	0.3548 (0.1769)	0.3568 (0.1712)	0.1840 (0.0561)
				3	0.1716 (0.0296)	0.6047 (0.4438)	0.5697 (0.3720)	0.5043 (0.2722)
			2	2	0.2047 (0.0426)	0.3495 (0.1792)	0.5174 (0.3461)	0.1932 (0.0635)
				3	0.3370 (0.1157)	0.5407 (0.4096)	0.4154 (0.2412)	0.4536 (0.2310)
		8	1	2	0.0977 (0.0096)	0.3300 (0.1576)	0.3677 (0.1774)	0.1746 (0.0527)
				3	0.1722 (0.0298)	0.6235 (0.4697)	0.5659 (0.3653)	0.5029 (0.2700)
			2	2	0.2036 (0.0421)	0.3589 (0.1818)	0.4909 (0.3198)	0.1973 (0.0644)
				3	0.3376 (0.1159)	0.5335 (0.3915)	0.3964 (0.2244)	0.4428 (0.2195)
	15	8	1	2	0.0906 (0.0082)	0.3199 (0.1485)	0.3838 (0.1973)	0.1956 (0.0605)
				3	0.1654 (0.0274)	0.6017 (0.4453)	0.5368 (0.3355)	0.4871 (0.2565)
			2	2	0.1975 (0.0396)	0.3585 (0.1858)	0.5026 (0.3333)	0.1969 (0.0648)
				3	0.3258 (0.1079)	0.4892 (0.3471)	0.3697 (0.2047)	0.4257 (0.2073)
		11	1	2	0.0901 (0.0081)	0.3215 (0.1491)	0.3830 (0.1948)	0.1937 (0.0602)
				3	0.1595 (0.0255)	0.5595 (0.3864)	0.4924 (0.2894)	0.4601 (0.2314)
			2	2	0.1934 (0.0379)	0.3375 (0.1684)	0.4857 (0.3239)	0.1990 (0.0636)
				3	0.3202 (0.1041)	0.4793 (0.3372)	0.3397 (0.1864)	0.4126 (0.1990)
40	20	10	1	2	0.0974 (0.0095)	0.2980 (0.1286)	0.3639 (0.1825)	0.1888 (0.0564)
				3	0.1717 (0.0295)	0.5423 (0.3605)	0.4585 (0.2523)	0.4460 (0.2176)
			2	2	0.2028 (0.0415)	0.3066 (0.1403)	0.4452 (0.2768)	0.1984 (0.0633)
				3	0.3372 (0.1147)	0.5088 (0.3544)	0.3227 (0.1660)	0.3980 (0.1854)

						Bayes Estimations		
N	m	k	α	β	*Adams Estimations*	*Chara-Prior*	*Gamma Prior*	*Kernel Prior*
		15	1	2	0.0973 (0.0095)	0.2871 (0.1208)	0.3642 (0.1806)	0.1813 (0.0525)
				3	0.1716 (0.0295)	0.5414 (0.3601)	0.4684 (0.2664)	0.4517 (0.2250)
			2	2	0.2028 (0.0415)	0.2966 (0.1294)	0.4366 (0.2685)	0.1916 (0.0587)
				3	0.3352 (0.1133)	0.4871 (0.3283)	0.3115 (0.1575)	0.3888 (0.1769)
	30	15	1	2	0.0923 (0.0085)	0.2731 (0.1084)	0.3536 (0.1754)	0.1816 (0.0517)
				3	0.1647 (0.0272)	0.5114 (0.3260)	0.4244 (0.2234)	0.4240 (0.2006)
			2	2	0.1966 (0.0389)	0.2945 (0.1261)	0.4311 (0.2666)	0.1903 (0.0545)
				3	0.3269 (0.1077)	0.4652 (0.3039)	0.2892 (0.1451)	0.3737 (0.1686)
		23	1	2	0.0901 (0.0081)	0.2835 (0.1149)	0.3514 (0.1760)	0.1953 (0.0582)
				3	0.1588 (0.0253)	0.4820 (0.2965)	0.3829 (0.1859)	0.3961 (0.1771)
			2	2	0.1908 (0.0366)	0.2723 (0.1128)	0.3963 (0.2359)	0.1831 (0.0521)
				3	0.3164 (0.1009)	0.4134 (0.2521)	0.2475 (0.1135)	0.3402 (0.1437)
60	30	15	1	2	0.0976 (0.0095)	0.2735 (0.1083)	0.3458 (0.1693)	0.1883 (0.0543)
				3	0.1723 (0.0297)	0.4944 (0.3059)	0.3841 (0.1869)	0.4015 (0.1806)
			2	2	0.2031 (0.0415)	0.2703 (0.1080)	0.4011 (0.2349)	0.1833 (0.0520)
				3	0.3352 (0.1130)	0.4599 (0.2855)	0.2708 (0.1288)	0.3590 (0.1554)
		23	1	2	0.0975 (0.0095)	0.2658 (0.1053)	0.3345 (0.1637)	0.1827 (0.0511)
				3	0.1723 (0.0297)	0.5068 (0.3203)	0.4010 (0.2014)	0.4131 (0.1908)
			2	2	0.2035 (0.0416)	0.2834 (0.1164)	0.3857 (0.2218)	0.1871 (0.0556)
				3	0.3351 (0.1130)	0.4547 (0.2800)	0.2625 (0.1186)	0.3526 (0.1488)
	45	23	1	2	0.0898 (0.0081)	0.2398 (0.0873)	0.3015 (0.1360)	0.1729 (0.0466)
				3	0.1610 (0.0260)	0.4409 (0.2473)	0.3348 (0.1460)	0.3665 (0.1530)
			2	2	0.1935 (0.0376)	0.2521 (0.0929)	0.3483 (0.1898)	0.1746 (0.0466)
				3	0.3208 (0.1034)	0.4141 (0.2405)	0.2398 (0.1088)	0.3300 (0.1383)
		34	1	2	0.0899 (0.0081)	0.2424 (0.0876)	0.3220 (0.1536)	0.1750 (0.0471)
				3	0.1591 (0.0253)	0.4323 (0.2358)	0.3291 (0.1444)	0.3621 (0.1512)
			2	2	0.1917 (0.0369)	0.2489 (0.0924)	0.3483 (0.1892)	0.1750 (0.0474)
				3	0.3175 (0.1014)	0.3953 (0.2256)	0.2256 (0.0914)	0.3172 (0.1271)